ESTIMATING A TRANSFORMATION BETWEEN PROJECTIONS USING LEAST-SQUARES

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One of the characteristics that has made SAS such a valuable tool in many data processing environments is its versatility. The breadth of coverage of statistical/mathmatical/graphical techniques allows the analyst to solve a wide variety of problems using a single product. Alternatively, a single problem using techniques from several areas may be solved entirely within SAS. The present paper illustrates a case where a moderately complex problem in mathematics/ statistics had to be solved in order to produce a map which was the original goal of the project.

Recently the U. S. Army Corps of Engineers (COE) approached us with the following requirement: They wanted to produce a set of maps showing all of the 535 COE communication node locations. The location of each node was known only in terms of the telephone number and name of the site. Could we find a way to translate telephone numbers into some form that would allow computer-mapping techniques to be employed?

Most SAS users are familiar with the tools provided by SAS for mapping. In particular, the U. S. state boundaries are available in latitude/longitude (unprojected) form, as well as a procedure (GPROJECT) for projecting the coordinates into an X/Y Cartesian form that PROC GMAP can use for plotting. Thus, if we could determine the latitude and longitude or the appropriate X/Y coordinates for each network location, we could create artificial "states" as, for example, small triangles, at each appropriate location, and then project and plot these "states" along with the standard state geographic boundaries using GPROJECT and GMAP.

A little research led us to discover that American Telephone and Telegraph maintains and distributes a table that relates telephone numbers (area codes and exchanges) to geographic location. In addition, the Network Services group of Boeing Computer Services has developed a system for automatically looking up locations by exchange and area code, and reporting back the corresponding geographic location coordinates. Unfortunately, geographic location in this context is represented on a specially-developed projection plane used by the telephone companies to compute long distance charges. This planar projection system, called the "V and H" code, is actually a Donald elliptic projection of the latitude and longitude, with the property that distances between points are maintained in the two-dimensional projection of the three-dimensional (Earth) surface. While this is useful for computing telephone charge tariffs, it is somewhat different from the more standard cartographic projections available in SAS (e.g., Alberts, Lambert, Gnomon). The standard projections maintain areas of polygons or angles at the corners of polygons equal between the original representation and the Cartesian projection, and produce maps of the sort we are used to seeing.

Thus, we could translate the COE phone locations into one Cartesian coordinate system, while SAS PROC GPROJECT would permit us to project the U.S. state boundaries using a different projection into a different Cartesian coordinate system. If the projections were not too different from one another, it seemed that it might be possible to find an algorithm to transform coordinates from one Cartesian system to another, i. e., to compute X and Y coordinates from V and H coordinates.

Since the two projection algorithms are not alike, the two Cartesian systems wouldn't necessarily fit together nicely. Thus, an approximate transformation would have to be computed that would minimize the topological stress in forcing the two surfaces together, and thus minimize the projection error. The "old reliable" of stress measures, least-squares, seem a reasonable criterion for assuring a good fit. What was required was an equation (or set of equations) that could be used in transforming one set of coordinates to another, using parameters estimated by some linear or non-linear regression model.

To understand how such a model might be approached, let us first consider some simple examples. Figure 1 illustrates a situation in which a single point, P, is located using two alternative Cartesian coordinate systems, one represented by axes X and Y, and the other by axes X1 and Y1. The scale on both systems is the same, but the origin has been moved. This is referred to as "axis translation." P is located at (6,4) in the X/Y system, and at (4,1) in the X1/Y1 system. Coordinates of one system may be transformed into those of the second by applying a system of two simple equations:

$X1 = X + TX$

$Y1 = Y + TY$ where TX and TY are the horizontal and vertical displacements respectively of the origin in the second system from the first. In Figure 1 TX is -2 and TY is -3.

A second basic kind of transformation is "axis scaling." In this situation there are again two alternative coordinate systems, with a common origin, but with different unit definitions on each axis. For example, the distance from 0 to 1 (1 unit) on the X axis might be equivalent to the distance from 0 to 4 (4 units) on the X1 axis; the X1 axis scale has been stretched with regard to the X axis. In this case coordinates of one system may be transformed in those of the second by applying two simple equations:

X1 = X * SX

Y1 = Y * SY where SX and SY are the horizontal and vertical scale factors (SX equals 4 in the example above).

Naturally, translation and scaling may be combined in a single pair of transformation equations:

X1 = (X + TX) * SX

Y1 = (Y + TY) * SY Figure 2 illustrates a pair of coordinate systems related by a translation/scaling transformation. Since the origin of X1/Y1 is at (2,3) on the X/Y system, TX is -2 and TY is -3. Since 1 unit on the X axis is equal to .5 units on the X1 axis, SX is .5; 1 unit on the Y axis correspond to 2 units on the Y1 axis, so SY is 2.0. The vertices of the plotted quadrilateral ABCD are at (4,4), (6,4), (6,5), and (4,5) on the X/Y system, and at (2,2), (4,2), (4,4), and (2,4) on the X1/Y1 system. For the second point of the quadrilateral, note that

2 = X1 = (6 - 2) * .5

2 = Y1 = (4 - 3) * 2.0 For any loci on these axes,

X1 = (X - 2) * .5

Y1 = (Y - 3) * 2.D

In addition to translation and scaling there is a third basic type of transformation, "axis rotation." With rotation, the second set of coordinate axes are rotated with respect to the first. In Figure 3, the X1/Y1 system has been rotated 45 degrees with respect to the X/Y system. In the case of rotation the following two equations are used to transform from one scheme to the other:

X1 = (X * Cos(A)) - (Y * Sin(A))

Y1 = (X * Sin(A)) + (Y * Cos(A)) where A is the angle between the X and X1 (or Y and Y1) axes. Note that the other two transformations, translation and scaling, each required two parameters each, while only one is required here. Since the axes must remain orthogonal to one another, the angle between one pair of axes (e. g. X and X1) must be the same as the angle between the other pair (Y and Y1). Note also that the angle A as a parameter yokes these two transformation equations together, where previously each of the equations in the system could be computed independently. Figure 3 illustrates the coordinate transformation for point P, at (3,3) on the X/Y coordinate system, and at (4.24,0) on the X1/Y1 system.

The following pair of equations may be used to describe the transformation between two coordinate systems that have been simultaneously translated, scaled, and rotated with respect to one simultaneously another:

X1 = TX + (SX * (((X * Cos(A)) - ((Y * Sin(A))))

Y1 = TY + (SY * (((X * Sin(A)) + ((Y * Cos(A)))) with transformation parameters TX, TY, SX, SY, and A.

In the present project, it was assumed that the "V & H" code axes could be related to the SAS/GRAPH projected axes by some combination of translation, scaling, and rotation. To estimate these transformation parameters, a sample of 44 COE communication nodes was selected. These locations were chosen systematically in order to assure that the sample was geographically dispersed (East/West and North/South). The latitude and longitude of each of these locations was then determined from the Rand McNally International Atlas (1979), and a small triangular "state" was designed at each point using the referenced location as the center.(1) These 44 states were then concatenated to the STATES map provided by SAS Institute, and GPROJECT was used to produce an X/Y Cartesian projection using the ALBERS algorithm.

Next, SYSNLIN was used to estimate the transformation parameters for computing X/Y coordinates from "V & H" coordinates on the sample of 44. A two-equation non-linear system (see above) was estimated using the SAS procedures MODEL and SYSNLIN. The resulting solution appeared to be quite satisfactory statistically, with a coefficient of determination for the X equation of 1.0000 and for the Y equation of .9998. All parameter estimates were significant by the t-statistic at beyond the .0001 probability level. The only area of concern was a moderate amount of intercorrelation among the estimated parameters (as might be expected). This intercorrelation was primarily between the translation and scaling parameters (TX and SX, correlation of -.80; TY and SY, correlation of -.86), but also included the rotation parameter (correlation between A and TX was -.58, and between A and TY .57).

Using the regression equations obtained from SYSNLIN, point "V & H" estimates were transformed to the SAS X/Y plane, and small square "states" were constructed at each estimated location. The actual and estimated locations of the 44 sampled points were then plotted on a US map using GMAP. The results are provided in Figure 4. It is apparent that the estimates are quite acceptable, at least on a map of this scale. The remaining 491 points were then transformed from "V & H" coordinates to the estimated X/Y plane using the equations estimated by SYSNLIN, and plotted to produce the final product, shown in Figure 5.

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(1) While the loci of interest were, for all practical purposes, points, GMAP can only plot closed polygons. Thus, each point locus was transformed into a small triangle for mapping.

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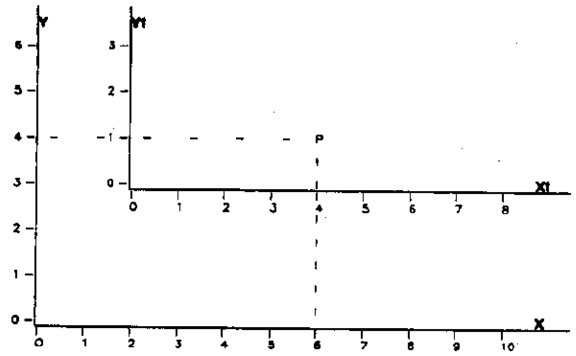

Figure 1. Axis Translation

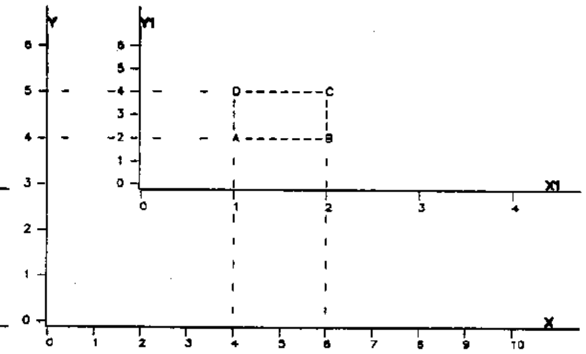

Figure 2. Axis Translation and Scaling

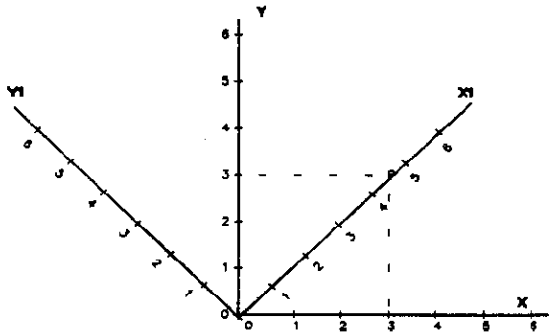

FIGURE 3. Axis Rotation

Triangles are real ---- Squares are estimated

Figure 4. Plot of 44 Test Locations

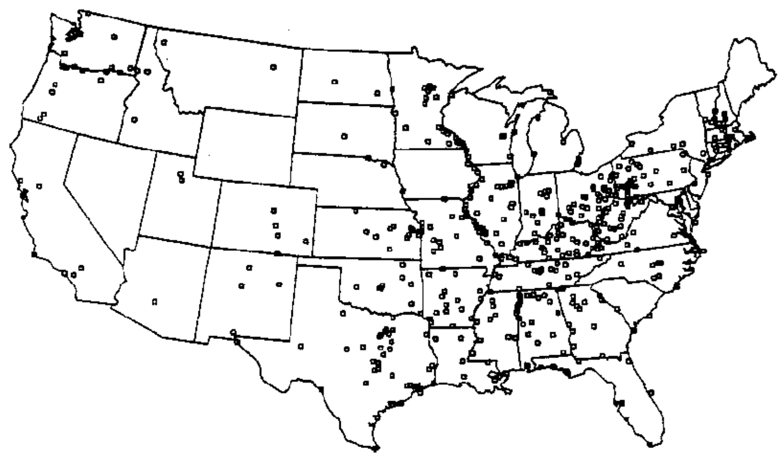

Figure 5. Corps of Engineers CNL's
(535 locations)